Exercise 12

In Exercises 11-14, (a) solve the given equation by the method of characteristic curves, and (b) check your answer by plugging it back into the equation.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Solution

Divide both sides by x.

$$\frac{\partial u}{\partial x} + \frac{y}{x}\frac{\partial u}{\partial y} = 0$$

The differential of a two-dimensional function $g = g(x, y)$ is given by

$$dg = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy.$$

Dividing both sides by dx yields the fundamental relationship between the total derivative of g and its partial derivatives.

$$\frac{dg}{dx} = \frac{\partial g}{\partial x} + \frac{dy}{dx}\frac{\partial g}{\partial y}$$

Comparing this to the PDE, we see that along the (characteristic) curves in the xy-plane defined by

$$\frac{dy}{dx} = \frac{y}{x} \tag{1}$$

the PDE reduces to the ODE,

$$\frac{du}{dx} = 0. \tag{2}$$

Solve equation (1), using ξ for the characteristic coordinate.

$$\frac{dy}{y} = \frac{dx}{x} \quad \rightarrow \quad \int \frac{dy}{y} = \int \frac{dx}{x} \quad \rightarrow \quad \ln|y| = \ln|x| + \xi \quad \rightarrow \quad \xi = \ln|y| - \ln|x| = \ln\left|\frac{y}{x}\right|$$

Then solve equation (2) by integrating both sides with respect to x.

$$u(x, \xi) = f(\xi)$$

Now that u is known, change back to the original variables.

$$u(x, y) = f\left(\ln\left|\frac{y}{x}\right|\right) = F\left(\frac{y}{x}\right)$$

Here f and F are arbitrary functions. Compute the first derivatives to check the solution.

$$\frac{\partial u}{\partial x} = F'\left(\frac{y}{x}\right) \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = F'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2}F'$$

$$\frac{\partial u}{\partial y} = F'\left(\frac{y}{x}\right) \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = F'\left(\frac{y}{x}\right) \cdot \left(\frac{1}{x}\right) = \frac{1}{x}F'$$

As a result,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{y}{x}F' + \frac{y}{x}F' = 0.$$